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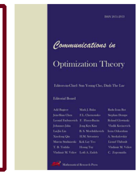

A NOTE ON THE UNIQUENESS OF ENTIRE FUNCTION $f(z)$ SHARING A SMALL FUNCTION WITH $f(qz)$

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Abstract. In this paper, we investigate the uniqueness problem related to an entire function $f(z)$ and its q-difference $f(qz)$, where q is a non-zero constant. We will prove some q-difference analogues of the theorem given by K.W. Yu [On entire and meromorphic functions that share small functions with their derivatives, J. Inequal. Pure Appl. Math. 4 (2003), Article ID 21].

Keywords. Entire function; Sharing value; Uniqueness.

1. Introduction-Results

In what follows, a meromorphic function will mean meromorphic in the whole complex plane. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM; if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM. Let m be a positive integer or infinity and $a \in C \cup \{\infty\}$. We denote by $E_{m)}(a,f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than m. We denote by $N_{k)}(r,1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted.

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Set

$$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+...+\overline{N}_{(k}(r,\frac{1}{f-a}).$$

As usual, by $S(r,f)$ we denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside of a possible exceptional set of finite linear measure. In particular, we denote by $S_1(r,f)$ any quantity satisfying $S_1(r,f)=o(T(r,f))$ for all r on a set of logarithmic density 1. It is assumed that the reader is familiar with the notations of Nevanlinna theory, that can be found, for instance, in [7] and [15]. Denote

$$\Theta(0,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,1/f)}{T(r,f)},$$

$$\delta_p(0,f)=1-\limsup_{r\to\infty}\frac{N_p(r,1/f)}{T(r,f)}.$$

We now explain in the following definition the notion of weighted sharing which was introduced by I. Lahiri [9].

Definition. [9] For a complex number $a\in C\cup\{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m\le k$ and $k+1$ times if $m>k$. For a complex number $a\in C\cup\{\infty\}$, such that $E_k(a,f)=E_k(a,g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\le k)$ and z_0 is a zero of $f-a$ with multiplicity $m(>k)$ if and only if it is a zero of $g-a$ with multiplicity $n(>k)$, where m is not necessarily equal to n. We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integer p, $0\le p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Rubel and Yang [12], Gundersen [5], Yang [13] and many other authors have obtained elegant results on the uniqueness problems of entire functions that share values CM or IM with their first or k-th derivatives. In the aspect of only one CM value, R.Brück posed the following conjecture.

Conjecture. [3] Let f be a nonconstant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite, if f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a}=c,$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

In 1998, Gundersen and Yang [6] proved that the conjecture is true if f is of finite order, and in 1999, Yang [14] generalized their result to the k-th derivatives. In 2004, Chen and Shon [4] proved that the conjecture is true for entire functions of first iterated order $\rho_1(f)<1/2$. In 2003, Yu considered the case that a is a small function and obtained the following result.

Theorem A. [17] *Let f be a nonconstant entire function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value* 0 *CM and $\delta(0, f) > 3/4$, then $f \equiv f^{(k)}$.*

Lin and Lin [10] improved Theorem B with the notion of weakly weighted sharing.

Theorem B. [10] *Let $k \geq 1$ and $2 \leq m \leq \infty$. Let f be a nonconstant entire function, and let a be a small function of f such that $a(z) \not\equiv 0, \infty$. If f and $f^{(k)}$ share "(a, m)" and $\delta_{2+k}(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

In 2010, Meng proved the following result.

Theorem C. [11] *Let f be a nonconstant entire function, and let a be a small function of f such that $a(z) \not\equiv 0, \infty$. If $\overline{E}_{4)}(a, f) = \overline{E}_{4)}(a, f^{(k)})$ and $E_{2)}(a, f) = E_{2)}(a, f^{(k)})$ and $\delta_{2+k}(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

Now one may ask the following question which is the motivation of the paper: Can we get q-difference analogues of the above results with the notion of weighed sharing ? Considering this question, we prove the following results.

Theorem 1. *Let f be a zero-order entire function, and $q \in C \backslash \{0\}$, $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f(z)$ and $f(qz)$ share $(a(z), 2)$ and $\Theta(0, f) > 3/4$, then $f \equiv f(qz)$.*

Theorem 2. *Let f be a zero-order entire function, and $q \in C \backslash \{0\}$, $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f(z)$ and $f(qz)$ share $(a(z), 1)$ and $\Theta(0, f) > 7/9$, then $f \equiv f(qz)$.*

Theorem 3. *Let f be a zero-order entire function, and $q \in C \backslash \{0\}$, $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f(z)$ and $f(qz)$ share $(a(z), 0)$ and $\Theta(0, f) > 6/7$, then $f \equiv f(qz)$.*

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

Lemma 1. [8] *Let F, G be two nonconstant meromorphic functions such that they share $(1, 2)$, and $H \not\equiv 0$. Then*

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G),$$

the same inequality holds for $T(r, G)$.

Lemma 2. [18] *Let f be a zero-order meromorphic function, and $q \in C \backslash \{0\}$. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

on a set of lower logarithmic density 1.

Lemma 3. [2] *Let f be a zero-order meromorphic function, and $q \in C\backslash\{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S_1(r,f).$$

Lemma 4. [1] *Let F, G be two nonconstant meromorphic functions such that they share $(1,1)$, and $H \not\equiv 0$. Then*

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G),$$

the same inequality holds for $T(r,G)$.

Lemma 5. [1] *Let F, G be two nonconstant meromorphic functions such that they share $(1,0)$, and $H \not\equiv 0$. Then*

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right) + 2\overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,F) + S(r,G),$$

the same inequality holds for $T(r,G)$.

3. Proof of Theorem 1

Let

$$F = \frac{f}{a}, \qquad G = \frac{f(qz)}{a}. \tag{3.1}$$

Then it is easy to verify F and G share $(1,2)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 1 that

$$T(r,F) + T(r,G) \le 2\left\{N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right)\right\} + S(r,F) + S(r,G),$$

that is,

$$T(r,f) + T(r,f(qz)) \le 2N_2\left(r,\frac{1}{f}\right) + 2N_2\left(r,\frac{1}{f(qz)}\right) + S(r,f). \tag{3.2}$$

Furthermore, we note that

$$N_2\left(r,\frac{1}{f}\right) \le 2\overline{N}\left(r,\frac{1}{f}\right), N_2\left(r,\frac{1}{f(qz)}\right) \le 2\overline{N}\left(r,\frac{1}{f}\right). \tag{3.3}$$

Using Lemma 2 and (3.2) (3.3), we obtain

$$T(r,f) \le 4\overline{N}\left(r,\frac{1}{f}\right) + S_1(r,f). \tag{3.4}$$

It follows that $4\Theta(0,f) \le 3$, which contradicts $\Theta(0,f) > 3/4$. Therefore $H \equiv 0$. That is

$$\frac{F''}{F'} - 2\frac{F'}{F-1} \equiv \frac{G''}{G'} - 2\frac{G'}{G-1}. \tag{3.5}$$

It follows that

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.6}$$

where $A(\neq 0)$ and B are constants. Therefore,

$$F = \frac{(B+1)G + (A-B-1)}{BG + (A-B)}. \tag{3.7}$$

Now we distinguish the following three cases.

Case 1. Suppose that $B \neq -1, 0$. If $A-B-1 \neq 0$, then from (3.7), we have

$$\overline{N}\left(r, \frac{1}{G + \frac{A-B-1}{B+1}}\right) = \overline{N}\left(r, \frac{1}{F}\right). \tag{3.8}$$

By the Second Fundamental Theorem, we have

$$T(r,G) < \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G + \frac{A-B-1}{B+1}}\right) + S(r,G), \tag{3.9}$$

that is,

$$T(r, f(qz)) < \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f), \tag{3.10}$$

and so

$$T(r,f) < 2\overline{N}\left(r, \frac{1}{f}\right) + S_1(r,f). \tag{3.11}$$

It follows that $\Theta(0,f) \leq 1/2$, which contradicts $\Theta(0,f) > 3/4$. Therefore $A-B-1=0$. From (3.7), we obtain

$$\overline{N}\left(r, \frac{1}{G + \frac{1}{B}}\right) = \overline{N}(r,F). \tag{3.12}$$

Similar to the arguments in the above, we also have a contradiction.

Case 2. Suppose that $B=-1$. If $A+1 \neq 0$. Then from (3.7), we have

$$\overline{N}\left(r, \frac{1}{G-(A+1)}\right) = \overline{N}(r,F). \tag{3.13}$$

Similar to the arguments in Case 1, we can get a contradiction. Therefore, $A+1=0$, then from (3.7), we have $FG \equiv 1$. From (3.1), we have

$$f(z)f(qz) \equiv a^2. \tag{3.14}$$

From (3.14) and Lemma 3, we obtain that

$$\begin{aligned} 2T(r, f(z)) = T\left(r, \frac{1}{f(z)^2}\right) + O(1) &= T\left(r, \frac{1}{f(z)} \frac{f(qz)}{a^2}\right) + O(1) \\ &= m\left(r, \frac{f(qz)}{f(z)}\right) + N\left(r, \frac{f(qz)}{f(z)}\right) + S_1(r,f) \\ &\leq T(r, f(z)) + S_1(r,f). \end{aligned}$$

Thus $T(r,f(z)) = S_1(r,f)$, which is impossible.

Case 3. Suppose that $B=0$. If $A-1 \neq 0$, then from (3.7), we have

$$\overline{N}\left(r, \frac{1}{G+(A-1)}\right) = \overline{N}\left(r, \frac{1}{F}\right). \tag{3.15}$$

Similar to the arguments in Case 1, we also have a contradiction. Therefore $A-1=0$. From (3.7) we have $F \equiv G$, this implies $f(z) \equiv f(qz)$. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let

$$F=\frac{f}{a},\qquad G=\frac{f(qz)}{a}. \tag{4.1}$$

Then it is easy to verify F and G share $(1,1)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 4 that

$$T(r,F)+T(r,G)\leq 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\right\}+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G). \tag{4.2}$$

Using Lemma 2 and (3.3) (4.2), we obtain

$$T(r,f)\leq \frac{9}{2}\overline{N}\left(r,\frac{1}{f}\right)+S_1(r,f). \tag{4.3}$$

It follows that $\Theta(0,f)\leq\frac{7}{9}$, which contradicts $\Theta(0,f)>7/9$. Therefore $H\equiv 0$. Similar to the arguments in Theorem 1, we see that Theorem 2 holds.

5. Proof of Theorem 3

Let

$$F=\frac{f}{a},\qquad G=\frac{f(qz)}{a}. \tag{5.1}$$

Then it is easy to verify F and G share $(1,0)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 5 that

$$T(r,F)+T(r,G)\leq 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\right\}+3\overline{N}\left(r,\frac{1}{F}\right)+3\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G). \tag{5.2}$$

Using Lemma 2 and (3.3) (5.2), we obtain

$$T(r,f)\leq 7\overline{N}\left(r,\frac{1}{f}\right)+S_1(r,f). \tag{5.3}$$

It follows that $\Theta(0,f)\leq\frac{6}{7}$, which contradicts $\Theta(0,f)>6/7$. Therefore $H\equiv 0$. Similar to the arguments in Theorem 1, we see that Theorem 3 holds.

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